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Rotations of a near-symmetrical satellite in an elliptical orbit with Mercury-type resonance☆

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ABSTRACT

The motion of a satellite, i.e., a rigid body, about to the centre of mass under the action of the gravitational moments of a central Newtonian gravitational field in an elliptical orbit of arbitrary eccentricity is investigated. It is assumed that the satellite is almost dynamically symmetrical. Plane periodic motions for which the ratio of the average value of the absolute angular velocity of the satellite to the average motion of its centre of mass is equal to 3/2 (Mercury-type resonance) are examined. An analytic solution of the non-linear problem of the existence of such motions and their stability to plane perturbations is given. In the special case in which the central ellipsoid of inertia of the satellite is almost spherical, the stability to spatial perturbations is also examined, but only in a linear approximation. ©2008.

1. Introduction

Consider the motion of a satellite about to the centre of mass in a central Newtonian gravitational field. We will treat the satellite as an absolutely rigid body, and we will assume that its linear dimensions are small compared with the characteristic size of the orbit. This enables us to assume[1] that the motion of the satellite about to the centre of mass does not influence the motion of the centre of mass itself. We will assume that the centre of mass of the satellite moves in an elliptical orbit. Let e denote the eccentricity of the orbit ($0 \le e < 1$), and let ν and M denote the true and mean anomalies, respectively. Then (see, for example, Ref. 2)

$$\frac{d\nu}{dt} = \frac{\omega_0}{(1-e^2)^{3/2}}\zeta^2, \quad M = \omega_0(t-t_0), \quad \omega_0 = \frac{2\pi}{\tau}, \quad \zeta = 1 + e\cos\nu \tag{1.1}$$

where ω_0 is the averge motion of the centre of mass, τ is the period of revolution of the satellite in its orbit, and t_0 is the time when the centre of mass of the satellite passes through the pericentre.

We will measure the motion of the satellite relative to an orbital system of coordinates with its origin at the centre of mass O of the satellite. Its OZ axis is directed along the radius vector of the centre of mass of the satellite relative to the attracting centre F, and the OX and OY axes are directed along the transversal and along the normal to the orbital plane, respectively. The $Oxyz$ system of coordinates that is rigidly connected to the satellite is formed by the principal central axes of inertia of the satellite. We will specify the orientation of the satellite relative to the orbital system of coordinates using the Euler angles ψ, θ and φ (Fig. 1).

Let A, B and C be the moments of inertia of the satellite about to the Ox, Oy and Oz axes. Using well-known[1] expressions for the force function and the kinetic energy of a satellite, we can find the Hamiltonian function Γ for the problem of the motion of a satellite about to the centre of mass under consideration. If we use p_ψ, p_θ and p_φ to denote the moment, a, made dimensionless using the multiplier

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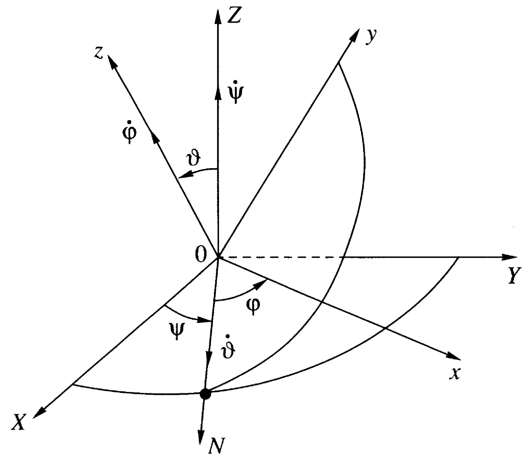

Fig. 1.

$A\omega_0(1-e^2)^{-3/2}$, and we use the first relation in (1.1) to change from t to the independent variable ν, we obtain

$$\Gamma = \frac{A\cos^2\varphi + B\sin^2\varphi}{2B\zeta^2\sin^2\theta}(p_\psi - p_\varphi\cos\theta)^2 + \frac{A\sin^2\varphi + B\cos^2\varphi}{2B\zeta^2}p_\theta^2 +$$
$$+\frac{A}{2C\zeta^2}p_\varphi^2 + \frac{(B-A)\sin 2\varphi}{2B\zeta^2\sin\theta}p_\theta(p_\psi - p_\varphi\cos\theta) - \cos\psi\,\mathrm{ctg}\,\theta\, p_\psi - \sin\psi\, p_\theta +$$
$$+\frac{\cos\psi}{\sin\theta}p_\varphi + \frac{3}{2A}\zeta[(B-A)\sin^2\theta\cos^2\varphi + (C-A)\cos^2\theta] \qquad (1.2)$$

The equations of motion allow of particular solutions that correspond to plane motions of the satellite when one of its principal axes of inertia (for example, Oz) is perpendicular to the plane of the orbit and the other two move in the orbital plane (Fig. 2). For plane motions

$$\theta = \pi/2, \quad \psi = \pi, \quad p_\theta = 0, \quad p_\psi = 0$$

and the variables φ and p_φ satisfy the canonical equations with Hamiltonian functions

$$G = \frac{A}{2C(1+e\cos\nu)^2}p_\varphi^2 - p_\varphi + \frac{3(B-A)}{4A}(1+e\cos\nu)\cos 2\varphi \qquad (1.3)$$

The plane resonant motions of a satellite were previously considered[3] using the method of averaging.

Mercury-type motion of a satellite is characterized by the fact (see Ref. 4) that when the centre of mass of the satellite passes through the pericentre of the orbit, one of the principal central axes of inertia of the satellite is directed along the radius vector of the centre of mass, whereas when the satellite passes through the apocentre, the same axis is perpendicular to the radius vector. Under these conditions, in a time equal to two periods of revolution of the centre of mass in the orbit, the satellite completes three turns about the normal to the orbital plane in absolute space (3:2 Mercury resonance).

The problem of the existence and stability of periodic motions of a satellite in the neighbourhood of its plane Mercury-type rotations in an elliptical orbit was first investigated numerically in a first approximation in Ref. 4. This investigation was continued in Ref. 5, and a broader range of admissible values of the problem parameters was considered. The non-linear problem of the existence and stability of periodic rotations of a Mercury-type satellite in the case of small eccentricities of the orbit was investigated analytically in Ref. 6.

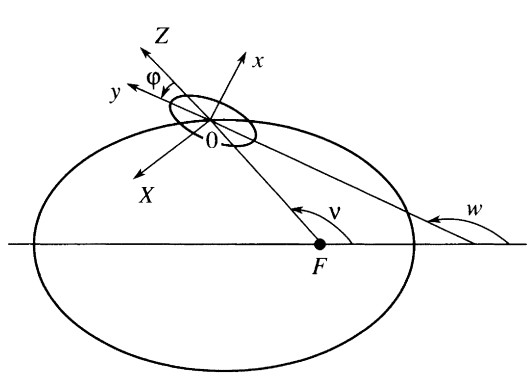

Fig. 2.

This paper presents an analytical investigation of the non-linear problem of the existence and stability of plane periodic rotations of a Mercury-type satellite for a satellite that is almost dynamically symmetrical (Section 2). Analytical and qualitative methods from the perturbation theory of Hamiltonian systems[7–9] are used. The linear problem of the stability of plane periodic Mercury-type rotations to spatial perturbations is also investigated under the assumption that the central ellipsoid of inertia of the satellite is nearly spherical (Section 3).

2. The existence of periodic Mercury-type rotations and their stability to plane perturbations

Instead of the variables φ and p_φ, we will introduce the new canonically conjugate variables w and I, which are more convenient for the ensuing analysis of the resonant periodic rotations of an almost symmetrical satellite. For this purpose, we will specify the generating function $S(I, \varphi, \nu)$ and the valence c of the transformation $\varphi, p_\varphi \to w, I$ according to the formulae

$$S = I(\varphi + \nu), \quad c = \frac{A}{C(1-e^2)^{3/2}}$$

We will use H to denote the new Hamiltonian function. The well-known relations[10]

$$w = \frac{\partial S}{\partial I} = \varphi + \nu, \quad cp_\varphi = \frac{\partial S}{\partial \varphi}, \quad H = cG + \frac{\partial S}{\partial \nu} \tag{2.1}$$

define the relation between the old and new variables and specify the expression for H. Note that w is the angle between the Oy axis of inertia of the satellite and the major axis of the orbit of its centre of mass (See Fig. 2).

If, based on relation (1.1), we change to a new independent variable, viz., the mean anomaly M, and introduce the small parameter $\varepsilon = |\gamma|/4$, where $\gamma = 3(A - B)/C$, we find that the following Hamiltonian function will correspond to the differential equations of the plane motion of the satellite in the variables w, I and M

$$H = H_0 + \varepsilon H_1 \tag{2.2}$$

where

$$H_0 = \frac{1}{2}I^2, \quad H_1 = -\sigma\frac{\zeta^3}{(1-e^2)^3}\cos(2w - 2\nu), \quad \sigma = \operatorname{sign}\gamma \tag{2.3}$$

In unperturbed motion (when $\varepsilon = 0$), I is constant, and

$$w = \omega M + Q \tag{2.4}$$

where Q is an arbitrary constant and ω is defined by the equality

$$\omega = \frac{\partial H_0}{\partial I} = I \tag{2.5}$$

The existence of periodic rotations. We will use the previously proposed algorithm[9] to analyse the perturbed system (when $0 < \varepsilon \ll 1$). When this algorithm is employed, an expression for the mean value $\bar{H}_1$ of the function H_1 on the unperturbed motion (2.4) is needed. Noting that, if the value of ω is not an integer or a half-integer, the function $\bar{H}_1$ is identically equal to zero, we will assume that

$$2\omega = m \tag{2.6}$$

where m is an integer. In this case

$$\overline{H_1} = -\sigma\Phi_m(e)\cos 2Q \tag{2.7}$$

where

$$\Phi_m(e) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\zeta^3}{(1-e^2)^3}\cos(mM - 2\nu)dM \tag{2.8}$$

The functions (2.8) were introduced into the theory of the motion of satellites about to their centre of mass by Chernous'ko.[3] They were subsequently studied in numerous papers.[4,11–13] A thorough analysis of the function (2.7) was carried out[14,15] with emphasis on the behaviour of these functions at values of e close to unity.

For Mercury-type rotations, the integer m in resonance relation (2.6) is equal to 3. The function $\Phi_3(e)$ can be represented in the form of the following series (see Section 4 below)

$$\Phi_3(e) = \frac{1}{3e^2}\sum_{n=-\infty}^{+\infty} a_n J_n(3e) z^{|n-3|}$$

$$a_n = |n-3|[(n-2)(n-4) + 6\sqrt{1-e^2}] - 3, \quad z = (1 - \sqrt{1-e^2})/e \tag{2.9}$$

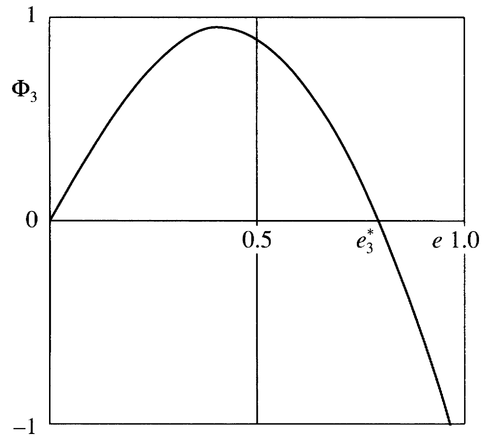

Fig. 3.

where J_n is the Bessel function of the first kind of order n.We will point out several properties of $\Phi_3(e)$, which are not difficult to establish on the basis of its analytic representation (2.9). We have $\Phi_3(0) = 0$, and for small e the function $\Phi_3(e)$ can be represented in the form of the series

$$\Phi_3(e) = \frac{7}{2}e - \frac{123}{16}e^3 + \frac{489}{128}e^5 - \frac{1763}{2048}e^7 + \frac{13527}{163840}e^9 - \frac{180369}{6553600}e^{11} - \frac{5986093}{367001600}e^{13} + \dots$$

In the interval (0, 1) the function $\Phi_3(e)$ has one root $e = e_3^* = 0.7881685123$, It is positive for $0 < e < e_3^*$ and negative for $e_3^* < e < 1$. A graph of the function $\Phi_3(e)$ is presented in Fig. 3. The maximum value of the function is reached when at $e = 0.4195582973$ and is equal to 0.9484247987. At $e = 1$ the graph has a vertical tangent, and $\Phi_3(1) = -1.4887724919$.

Let $e \neq e_3^*$. Then $\Phi_3(e) \neq 0$, and the equation $\partial \bar{H}_1/\partial Q = 0$ is equivalent to the equation $\sin 2Q = 0$. The latter equation has the roots $Q = s\pi/2$, where $s = 0, 1, 2, 3$ (other integer values of s do not give mechanically different solutions). In this case

$$\frac{\partial^2 \overline{H_1}}{\partial Q^2} = 4(-1)^s \sigma \Phi_3(e) \neq 0 \tag{2.10}$$

Consequently,[9] for sufficiently small ε, there are four plane periodic rotations of the satellite which are analytic in ε, and when $\varepsilon = 0$ they become periodic rotations, for which

$$w = \frac{3}{2}M + \frac{s\pi}{2}, \quad s = 0, 1, 2, 3 \tag{2.11}$$

We will call rotations for which $Q = 0$ or $Q = \pi$ (i.e., $s = 0$ or $s = 2$) rotations of type $Q^{(0)}$, and we will call rotations for which $Q = \pi/2$ or $Q = 3\pi/2$ (i.e., $s = 1$ or $s = 3$) rotations of type $Q^{(1)}$. For rotations of type $Q^{(0)}(Q^{(1)})$, the axis of inertia $Ox(Oy)$ of the satellite is perpendicular to the major axis of the orbit at the pericentre.

Explicit expressions for periodic rotations in the first approximation with respect to ε. The equations of motion that correspond to the Hamiltonian function (2.2) have the form

$$\frac{dw}{dM} = I, \quad \frac{dI}{dM} = -2\varepsilon\sigma\frac{\zeta^3}{(1-e^2)^3}\sin(2w - 2\nu) \tag{2.12}$$

We set

$$w = \frac{3}{2}M + s\frac{\pi}{2} + \varepsilon w^{(1)} + \dots, \quad I = \frac{3}{2} + \varepsilon I^{(1)} + \dots, \quad s = 0, 1, 2, 3 \tag{2.13}$$

After substituting expansions (2.13) into equalities (2.12) and equating the terms of the first power in ε on their right- and left-hand sides, we obtain a system of differential equations for finding the functions $w^{(1)}$ and $I^{(1)}$ that are 2π-periodic in M (or in ν, which is equivalent)

$$\frac{dw^{(1)}}{dM} = I^{(1)}, \quad \frac{dI^{(1)}}{dM} = -2(-1)^s\sigma\frac{\zeta^3}{(1-e^2)^3}\sin(3M - 2\nu) \tag{2.14}$$

Let

$$f_1(\nu) = \int_0^\nu \zeta\sin(3M - 2\nu)d\nu, \quad f_2(\nu) = \int_0^\nu \frac{f_1(\nu)}{\zeta^2}d\nu \tag{2.15}$$

If we introduce the notation $c_1 = f_2(2\pi)/(2\pi)$, the 2π-periodic solutions of system of Eq. (2.14) sought can be written in the form

$$w^{(1)} = 2(-1)^s\sigma(c_1 M - f_2(\nu)), \quad I^{(1)} = 2(-1)^s\sigma[c_1 - f_1(\nu)(1-e^2)^{-3/2}] \tag{2.16}$$

where M is a function of ν, defined by relations (1.1). The arbitrary additive constant in the expression for $w^{(1)}$ is taken equal to zero in order to satisfy the requirement $w^{(1)}(0)=0$ (the condition that at the pericentre of the orbit one of the principal central axes of inertia of the satellite is directed along the radius vector of its centre of mass).

From equalities (2.1), (2.13) and (2.16) we find the following expressions for periodic rotations of a satellite in the original variables φ and p_φ

$$\varphi = \frac{3}{2}M - \nu + s\frac{\pi}{2} + (-1)^s\frac{3(A-B)}{2C}(c_1 M - f_2(\nu)) + O(\varepsilon^2)$$
$$p_\varphi = \frac{3C(1-e^2)^{3/2}}{2A} + (-1)^s\frac{3(A-B)}{2A}[c_1(1-e^2)^{3/2} - f_1(\nu)] + O(\varepsilon^2)$$
$$s = 0,1,2,3 \tag{2.17}$$

The stability of periodic rotations. Let us consider the stability of the plane periodic motions (2.17) to plane perturbations. For both motions of type $Q^{(0)}$ (when $s=0$ or $s=2$ in equalities (2.17)) and motions of type $Q^{(1)}$ (when $s=1$ or $s=3$) we have

$$5\left(\frac{\partial^3\overline{H_1}}{\partial Q^3}\right)^2 - 3\frac{\partial^2\overline{H_1}}{\partial Q^2}\frac{\partial^4\overline{H_1}}{\partial Q^4} = 192\Phi_3^2(e) \neq 0$$

Also taking into account the fact that $\partial^2 H_0/\partial I^2 = 1 > 0$, we find[9] that if $\partial^2\bar{H}_1/\partial Q^2 > 0$, the periodic motion is unstable, while if $\partial^2\bar{H}_1/\partial Q^2 < 0$, Lyapunov stability exists. Taking into account equality (2.10), we hence find that, for sufficiently small ε, motions of type $Q^{(0)}$ are Lyapunov stable if $\gamma\Phi_3(e)>0$ and are unstable if $\gamma\Phi_3(e)<0$. Conversely, motions of type $Q^{(1)}$ are stable if $\gamma\Phi_3(e)<0$ and are unstable if $\gamma\Phi_3(e)>0$.

When the aforementioned properties of the function $\Phi_3(e)$ are taken into account (see also Fig. 3), for stable periodic Mercury-type rotation we find that the axis of the smallest of the moments of inertia of the satellite (A or B) is directed along the radius vector of the centre of mass at the pericentre if $0 < e < e_3^* = 0.7881685126$ and is perpendicular to the radius vector if $e_3^* < e < 1$.

Note that the eccentricity of Mercury's orbit ($e=0.2056$) corresponds to the first of these cases.

3. The stability of Mercury-type rotations to spatial perturbations

Consider the periodic rotation of a Mercury-type satellite specified by formulae (2.17), in which $s=0$. We will assume that $A>B$. As was noted in Section 2, when only plane perturbations are present, the motion is stable if $0 < e < e_3^*$ and unstable if $e_3^* < e < 1$. We will assume that the former case occurs, i.e., $0 < e < e_3^*$, and we will investigate the stability of Mercury-type rotation to spatial perturbations. In this process we will confine ourselves analysing the stability in the first (linear) approximation. We will assume that the central ellipsoid of inertia of the satellite is nearly spherical.

Investigative method. In function (1.2) let

$$\theta = \pi/2 + q_1, \quad \psi = \pi + q_2, \quad p_\theta = p_1, \quad p_\psi = p_2$$

The quadratic part Γ_2 of the expansion of this function in a power series in q_i and p_i ($i=1, 2$) describes the linear problem of the stability of plane motions of the satellite to spatial perturbations. It has the form

$$\Gamma_2 = \frac{1}{2}\left(\frac{A\cos^2\varphi + B\sin^2\varphi}{B\zeta^2}p_\varphi^2 - p_\varphi - 3\zeta\frac{A\sin^2\varphi + B\cos^2\varphi - C}{A}\right)q_1^2 +$$
$$+ \frac{(B-A)\sin 2\varphi}{2B\zeta^2}p_\varphi q_1 p_1 + \left(\frac{A\cos^2\varphi + B\sin^2\varphi}{B\zeta^2}p_\varphi - 1\right)q_1 p_2 +$$
$$+ \frac{1}{2}p_\varphi q_2^2 + q_2 p_1 + \frac{A\sin^2\varphi + B\cos^2\varphi}{2B\zeta^2}p_1^2 +$$
$$+ \frac{(B-A)\sin 2\varphi}{2B\zeta^2}p_1 p_2 + \frac{A\cos^2\varphi + B\sin^2\varphi}{2B\zeta^2}p_2^2 \tag{3.1}$$

The quantities φ and p_φ correspond to unperturbed plane motion.

We introduce two dimensionless inertial parameters using the formulae

$$\delta = \frac{A-B}{A}, \quad \kappa = \frac{A-C}{A-B} \tag{3.2}$$

Then

$$B = A(1-\delta), \quad C = A(1-\delta\kappa) \tag{3.3}$$

Since, according to the assumption made, $A>B$, it follows for A, B and C from the triangle inequalities that the region of admissible values of the parameters δ and κ is given by the inequalities

$$0 < \delta \le 1, \quad -1/\delta + 1 \le \kappa \le 1/\delta - 1$$

The fact that the ellipsoid of inertia is nearly spherical means that $0<\delta\ll 1$ and that the values of κ are such that if $\delta\to 0$,then $\delta\kappa\to 0$. We will henceforth regard δ as a small parameter of the problem.

When equalities (3.3) are taken into account, the unperturbed plane rotation of the satellite (2.17) is written in the following form (we recall that $s=0$)

$$\varphi = \frac{3}{2}M-\nu+\delta\frac{3}{2}(c_1M-f_2(\nu))+O(\delta^2),\quad p_\varphi = \frac{3}{2}(1-e^2)^{3/2}-\delta h(\nu)+O(\delta^2)$$
$$h(\nu) = \frac{3}{2}[(\kappa-c_1)(1-e^2)^{3/2}+f_1(\nu)] \tag{3.4}$$

Substituting expressions (3.3) and (3.4) into the right-hand side of formula (3.1) and expanding the result in a power series in δ, we obtain

$$\Gamma_2 = \Gamma_2^{(0)}+\delta\Gamma_2^{(1)}+O(\delta^2) \tag{3.5}$$

where

$$\Gamma_2^{(0)} = \frac{1}{2}g_0\left(\frac{g_0}{\zeta^2}-1\right)q_1^2+\left(\frac{g_0}{\zeta^2}-1\right)q_1p_2+\frac{1}{2}g_0q_2^2+q_2p_1+\frac{1}{2\zeta^2}(p_1^2+p_2^2)$$
$$g_0 = \frac{3}{2}(1-e^2)^{3/2} \tag{3.6}$$

$$\Gamma_2^{(1)} = \frac{1}{2}\left[\frac{g_0^2\cos^2\mu}{\zeta^2}-\left(\frac{2g_0}{\zeta^2}-1\right)h(\nu)+3\zeta(\cos^2\mu-\kappa)\right]q_1^2+\frac{g_0\sin 2\mu}{2\zeta^2}q_1p_1+$$
$$+\frac{g_0\cos^2\mu-h(\nu)}{\zeta^2}q_1p_2-\frac{1}{2}h(\nu)q_2^2+\frac{\sin^2\mu}{2\zeta^2}p_1^2+\frac{\sin 2\mu}{2\zeta^2}p_1p_2+\frac{\cos^2\mu}{2\zeta^2}p_2^2$$
$$\mu = \nu-\frac{3}{2}M \tag{3.7}$$

The stability of the system with the Hamiltonian function (3.5) can be analysed as in Ref. 16, where the steady of the rotation of a dynamically symmetrical satellite about a normal to the plane of an elliptic orbit was examined.

When $\delta=0$, we have $\Gamma_2\Gamma_2^{(0)}$. For the fundamental solution matrix of the system with Hamiltonian function $\Gamma_2\Gamma_2^{(0)}$, the following expression can be obtained

$$\mathbf{X}(\nu) = \left\|\begin{matrix}\cos\mu & \sin\nu & g_1 & -g_2\\ -\sin\mu & \cos\nu & g_2 & g_1\\ g_0\sin\mu & 0 & \cos\mu & \sin\mu\\ 0 & -g_0\sin\nu & -\sin\nu & \cos\nu\end{matrix}\right\| \tag{3.8}$$

Here

$$g_1 = \frac{\sin\nu-\sin\mu}{g_0},\quad g_2 = \frac{\cos\nu-\cos\mu}{g_0}$$

The value of μ is given by the last of the equalities in (3.7), and M is a function of ν defined by relations (1.1).

We can verify that the characteristic equation of the matrix $\mathbf{X}(2\pi)$ does not depend on e and has the form $(\rho^2-1)^2=0$ with two linearly independent eigenvectors corresponding to each of the twofold multipliers $\rho=\pm 1$. Consequently, the matrix $\mathbf{X}(2\pi)$ is reduced to into diagonal form, and when $\delta=0$, stability occurs.[17]

When $\delta=0$, the characteristic exponents will be pure imaginary numbers: $\pm i, \pm i/2$. Therefore, for small, but non-zero values of δ, twofold parametric resonance occurs for any value of e. An analysis of the stability when $0<\delta\ll 1$ can be performed based on transformation of the Hamiltonian function (3.5) by the Deprit–Hori method.[18] As was done previously,[16] we first make the canonical replacement of variables

$$\mathbf{u} = \mathbf{X}(\nu)\boldsymbol{\upsilon},\quad \mathbf{u}' = (q_1,q_2,p_1,p_2),\quad \boldsymbol{\upsilon}' = (x_1,x_2,X_1,X_2) \tag{3.9}$$

In the new variables, the Hamiltonian function (3.5) is written in the form $G=\delta G^{(1)}+O(\delta^2)$, where $G^{(1)}$ is the function $\Gamma_2^{(1)}$ (3.7), in which the replacement of variables (3.9) was made:

$$G^{(1)} = \sum g_{\nu_1\nu_2\mu_1\mu_2}x_1^{\nu_1}x_2^{\nu_2}X_1^{\mu_1}X_2^{\mu_2} \tag{3.10}$$

Here ν_1, ν_2, μ_1 and μ_2 are non-negative integers, whose sum is equal to 2.

Now, according to the algorithm of the Deprit–Hori method, we construct the canonical transformation

$$x_1,x_2,X_1,X_2\to y_1,y_2,Y_1,Y_2$$

which eliminates the independent variable ν from the new Hamiltonian in the terms that are of the first-order in δ. For the new Hamiltonian we obtain the expression $K = \delta K^{(1)} + O(\delta^2)$, where

$$K^{(1)} = \sum k_{\nu_1\nu_2\mu_1\mu_2} y_1^{\nu_1} y_2^{\nu_2} Y_1^{\mu_1} Y_2^{\mu_2} \tag{3.11}$$

and $k_{\nu_1\nu_2\mu_1\mu_2}$ are the values of the functions $g_{\nu_1\nu_2\mu_1\mu_2}$ averaged over ν. They are functions of the parameters e and κ.

After discarding terms higher than the first order in δ in the function K, we arrive at an approximate system described by a linear autonomous system of differential equations of the form

$$\frac{dy_j}{d\eta} = \frac{\partial K^{(1)}}{\partial Y_j}, \quad \frac{dY_j}{d\eta} = -\frac{\partial K^{(1)}}{\partial y_j}; \quad \eta = \delta\nu; \quad j = 1, 2 \tag{3.12}$$

Thus, to a first approximation in δ, the stability investigation problem reduces to an analysis of the characteristic equation of system (3.12)

$$\lambda^4 + a\lambda^2 + b = 0 \tag{3.13}$$

whose coefficients are functions of the parameters e and κ.

Analysis shows that the four coefficients $k_{1100}, k_{1010}, k_{0101}$ and k_{0011} of the function $K^{(1)}$ are identically equal to zero. Numerical integration is needed to obtain the remaining six coefficients in the general case. An analytical investigation is possible for small values of the eccentricity.

Results of the stability investigation. For small values of e, the coefficients of the quadratic form (3.11), which are not identically equal to zero, can be represented in the form of power series in e

$$k_{2000} = -\frac{3}{8}(\kappa - 1)\left(5 + 7e - \frac{9}{2}e^2 - \frac{207}{8}e^3 + \dots\right)$$

$$k_{1001} = -\frac{1}{2}(\kappa - 1)\left(5 + 7e + 3e^2 - \frac{123}{8}e^3 + \dots\right)$$

$$k_{0200} = -\frac{3}{8}(2\kappa - 1) + \frac{21}{16}e - \frac{195}{32}e^3 + \dots$$

$$k_{0110} = -\frac{1}{2}(2\kappa - 1) + \frac{7}{4}e - \frac{3}{4}(2\kappa - 1)e^2 - \frac{11}{2}e^3 + \dots$$

$$k_{0020} = -\frac{1}{6}(7\kappa - 1) + \frac{7}{12}(2\kappa + 1)e - \frac{1}{4}(11\kappa - 2)e^2 - \frac{1}{48}(39\kappa + 46)e^3 + \dots$$

$$k_{0002} = -\frac{1}{6}(7\kappa - 6) - \frac{7}{12}(2\kappa - 3)e - \frac{1}{4}(11\kappa - 9)e^2 + \frac{1}{48}(39\kappa - 32)e^3 + \dots$$

The coefficients of the characteristic Eq. (3.13) are represented by series of the form

$$a = \frac{1}{4}[29\kappa(\kappa - 1) + 1] - \frac{7}{4}(2\kappa - 1)e - \frac{1}{16}[28\kappa(\kappa - 1) - 61]e^2 + \frac{39}{32}(2\kappa - 1)e^3 + \dots \tag{3.14}$$

$$b = \frac{1}{16}\kappa(\kappa - 1)\left\{25(2\kappa - 1)^2 - 175(2\kappa - 1)e + \frac{7}{4}[128\kappa(\kappa - 1) + 207]e^2 + \right.$$
$$\left. + \frac{2039}{8}(2\kappa - 1)e^3 + \dots\right\} \tag{3.15}$$

and for the quantity $d = a^2 - 4b$ we have the following expansion

$$d = \frac{1}{16}[21\kappa(\kappa - 1) - 1]^2 + \frac{7}{8}(2\kappa - 1)[21\kappa(\kappa - 1) - 1]e - \frac{3}{32}(868\kappa^4 - 1736\kappa^3 +$$
$$+ 1123\kappa^2 - 255\kappa - 53)e^2 - \frac{5}{64}(2\kappa - 1)[511\kappa(\kappa - 1) + 163]e^3 + \dots \tag{3.16}$$

The quadratic form (3.11) is written in the form of the sum of the two quadratic forms $K_1^{(1)}$ and $K_2^{(1)}$, each of which depends on "its own" variables: $K_1^{(1)} = K_1^{(1)}(y_1, Y_2)$, $K_2^{(1)} = K_2^{(1)}(y_2, Y_1)$. When $e = 0$, we have

$$K_1^{(1)} = -\frac{15}{8}(\kappa - 1)y_1^2 - \frac{5}{2}(\kappa - 1)y_1Y_2 - \frac{1}{6}(7\kappa - 6)Y_2^2$$

$$K_2^{(1)} = -\frac{3}{8}(2\kappa - 1)y_2^2 - \frac{1}{2}(2\kappa - 1)y_2Y_1 - \frac{1}{6}(7\kappa - 1)Y_1^2$$

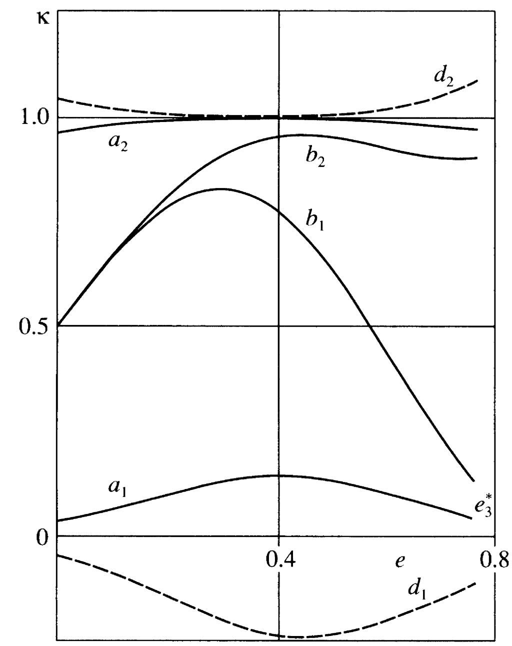

Fig. 4.

When $\kappa < 0$, both of these quadratic forms are positive definite, and when $\kappa > 1$, they are negative definite. Therefore, if $e = 0$, the quadratic form (3.11) is sign-definite when $\kappa < 0$ or $\kappa > 1$. This property is not lost when the first and higher powers of e are taken into account in the expansion of $K^{(1)}$ in a power series in e if e is sufficiently small.[17] Hence, By the Lyapunov stability theorem,[17] if $\kappa < 0$ or $\kappa > 1$, it follows that the plane rotations of the satellite under consideration are stable to spatial perturbations in a first approximation in δ for sufficiently small values of e.

If $0 < \kappa < 1$, instability occurs for small values of e. This follows from the fact that in this case the characteristic Eq. (3.13) has the root

$$\lambda = \sqrt{\kappa(1-\kappa)}\left(\frac{5}{2} - \frac{19}{20}e^2 + \ldots\right)$$

which is positive for sufficiently small values of e.

For values of e that are not small, the stability was investigated by numerically finding the coefficients in the characteristic Eq. (3.13) and then checking the stability conditions, which are given by the inequalities $a > 0$, $b > 0$ and $d > 0$ (if at least one of these inequalities holds with the reverse sign, instability occurs). The results of the numerical investigation are illustrated in Fig. 4.

The coefficient a in Eq. (3.13) vanishes on two curves: $\kappa = a_i(e)$ $(i = 1, 2)$. It follows from representation (3.14) that such expansions in power series in e are valid for the functions $a_i(e)$ $(i = 1, 2)$ for small values of e:

$$a_{1,2} = \frac{1}{2} \mp \frac{5}{58}\sqrt{29} + \frac{7}{29}e \pm \frac{1601}{16820}\sqrt{29}e^2 - \frac{739}{6728}e^3 + \ldots$$

For an assigned value $e \in [0, e_3^*]$ the coefficient a is positive if $\kappa < a_1(e)$ or $\kappa > a_2(e)$, and negative if $a_1(e) < \kappa < a_2(e)$.

When $\kappa = 0$ or $\kappa = 1$, the coefficient b is equal to zero for all $e \in [0, e_3^*]$. In addition, this coefficient vanishes on two curves: $\kappa = b_i(e)$ $(i = 1, 2)$. According to (3.15), for small values of e, the functions $b_i(e)$ $(i = 1, 2)$ can be expanded in series of the form

$$b_1 = \frac{1}{2} + \frac{7}{4}e - \frac{65}{8}e^3 + \ldots, \quad b_2 = \frac{1}{2} + \frac{7}{4}e - \frac{77}{16}e^3 + \ldots$$

The coefficient b is positive if either $\kappa < 0$ or $b_1(e) < \kappa < b_2(e)$ or $\kappa > 1$. If $0 < \kappa < b_1(e)$ or $b_2(e) < \kappa < 1$, then $b < 0$.

The quantity $d = a^2 - 4b$ is always positive except for values of the parameters e and κ that lie on the curves $\kappa = d_i(e)$ $(i = 1, 2)$, where d vanishes. Graphs of $\kappa = d_i(e)$ $(i = 1, 2)$ are shown as dashed lines in Fig. 4. For these graphs, from equality (3.16) we can obtain the expansions

$$d_{1,2} = \frac{1}{2} \mp \frac{5}{42}\sqrt{21} - \frac{1}{3}e \pm \frac{1601}{8820}\sqrt{21}e^2 - \frac{131}{504}e^3 + \ldots$$

It follows from the results of the calculations presented that if $0 < \kappa < 1$, instability occurs for any $e \in [0, e_3^*]$, because either a or b or both of these coefficients of the characteristic Eq. (3.13) are negative.

If $\kappa < 0$ or $\kappa > 1$, and the parameters e and κ do not lie on the graph of $\kappa = d_1(e)$ or $\kappa = d_2(e)$, the quantities a, b and d are positive, i.e., the stability conditions are satisfied. If the parameters e and κ lie on the curve $\kappa = d_1(e)$ or $\kappa = d_2(e)$, then $a > \theta$, $b > 0$, $d = 0$, and the characteristic Eq. (3.13) has two pairs of multiple, pure imaginary roots. The calculations show that the matrix of linear system (3.12) in that case is reduced to diagonal form; therefore, stability also occurs for values of e and κ that belong to the curves $\kappa = d_1(e)$ and $\kappa = d_2(e)$.

Thus, the conclusions regarding stability that were drawn for small e also hold for any value of e considered ($0 \le e \le e_3^* = 0.7881685126$): if $0 < \kappa < 1$ (i.e., $A > C > B$; in unperturbed plane motion, rotation of the satellite occurs about the axis of the moment of inertia of moderate magnitude), Mercury-type plane rotation of the satellite is unstable to spatial perturbations for fairly small values of δ, but if $\kappa < 0$ or $\kappa > 1$ (i.e., $C > A > B$ or $A > B > C$; unperturbed rotation occurs about the axis of the largest or smallest of the moments of inertia), stability occurs in a first approximation with respect to δ.

4. Analytical representation of the function $\Phi_m(e)$ using Bessel functions.

If we use relations (1.1) to change to integration with respect to ν on the right-hand side of equality (2.8), we obtain

$$\Phi_m(e) = \frac{1}{2\pi(1-e^2)^{3/2}}\int_0^{2\pi}\zeta\cos(mM-2\nu)d\nu \tag{4.1}$$

It is obvious that $\Phi_0(e) = 0$. We will calculate $\Phi_m(e)$ for the arbitrary integer m. To evaluate the integral on the right-hand side of equality (4.1), we use the well-known[2] relations

$$\frac{d\nu}{dE} = \frac{(1-e^2)^{1/2}}{1-e\cos E}, \quad \sin\nu = \frac{(1-e^2)^{1/2}\sin E}{1-e\cos E}, \quad \cos\nu = \frac{\cos E - e}{1-e\cos E} \tag{4.2}$$

and Kepler's equation

$$E - e\sin E = M \tag{4.3}$$

to change to a new variable, namely, the eccentric anomaly E.

We first rewrite equality (4.1) in the form

$$\Phi_m(e) = \frac{1}{2\pi(1-e^2)^{3/2}}\left[\int_0^{2\pi}\zeta\cos mM\cos 2\nu d\nu - \right.$$
$$\left. - \int_0^{2\pi}\sin mMd\left(\cos^2\nu + \frac{2}{3}e\cos^3\nu\right)\right]$$

After integrating by parts in the second integral and then performing elementary transformations using relations (4.2) and (4.3), we obtain the following expression for $\Phi_m(e)$

$$\Phi_m(e) = \frac{1}{2\pi e^2}\sum_{k=1}^{4}\int_0^{2\pi}\frac{\alpha_k\cos(mE - me\sin E)}{(1-e\cos E)^k}dE \tag{4.4}$$

where

$$\alpha_1 = -m(1-e^2)^{1/2}, \quad \alpha_2 = 2 - e^2 + \frac{2}{3}m(1-e^2)^{3/2}$$
$$\alpha_3 = -4(1-e^2), \quad \alpha_4 = 2(1-e^2)^2 \tag{4.5}$$

It follows from the well-known relations[19]

$$\cos(y\sin x) = \sum_{n=-\infty}^{+\infty}J_n(y)\cos nx, \quad \sin(y\sin x) = \sum_{n=-\infty}^{+\infty}J_n(y)\sin nx$$

where J_n is the Bessel function of the first kind of order n, that

$$\cos(mE - me\sin E) = \sum_{n=-\infty}^{+\infty}J_n(me)\cos(n-m)E \tag{4.6}$$

Substituting expression (4.6) into the right-hand side of equality (4.4), we obtain

$$\Phi_m(e) = \frac{1}{e^2}\sum_{n=-\infty}^{+\infty}J_n(me)\sum_{k=1}^{4}\alpha_k G_k \tag{4.7}$$

where we have introduced the notation

$$G_k = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(n-m)E}{(1-e\cos E)^k}dE, \quad k = 1, 2, 3, 4 \tag{4.8}$$

It is not difficult to prove that the following recurrence relation holds

$$G_{k+1} = G_k + \frac{e}{k}\frac{dG_k}{de}, \quad k \geq 1 \tag{4.9}$$

It is easy to calculate G_1 after applying the theory of residue's. We obtain

$$G_1 = \frac{z^{|n-m|}}{\sqrt{1-e^2}}, \quad z = \frac{1-\sqrt{1-e^2}}{e} \tag{4.10}$$

Relations (4.9) and (4.10) easily enable us to calculate G_2, G_3 and G_4. Then, after substituting the values of the G_k found into the right-hand side of equality (4.7) and performing some reduction taking notation (4.5) into account, we obtain the final expression for $\Phi_m(e)$:

$$\Phi_m(e) = \frac{1}{3e^2}\sum_{n=-\infty}^{+\infty} a_n(m,e)J_n(me)z^{|n-m|}$$

$$a_n = |n-m|[(n-m+1)(n-m-1)+2m\sqrt{1-e^2}]-m \tag{4.11}$$

Somewhat different forms of the analytical representation of the functions $\Phi_m(e)$ were previously obtained using Bessel functions.[13–15]

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